Oxford Cambridge and RSA

Thursday 25 May 2023 – Afternoon

A Level Further Mathematics B (MEI)

Y420/01 Core Pure

Time allowed: 2 hours 40 minutes

* 9 9 6 9 0 6 7 0 5 2 *

You must have:
- the Printed Answer Booklet
- the Formulae Booklet for Further Mathematics B (MEI)
- a scientific or graphical calculator

QP

INSTRUCTIONS
- Use black ink. You can use an HB pencil, but only for graphs and diagrams.
- Write your answer to each question in the space provided in the **Printed Answer Booklet**. If you need extra space use the lined pages at the end of the Printed Answer Booklet. The question numbers must be clearly shown.
- Fill in the boxes on the front of the Printed Answer Booklet.
- Answer **all** the questions.
- Where appropriate, your answer should be supported with working. Marks might be given for using a correct method, even if your answer is wrong.
- Give your final answers to a degree of accuracy that is appropriate to the context.
- Do **not** send this Question Paper for marking. Keep it in the centre or recycle it.

INFORMATION
- The total mark for this paper is **144**.
- The marks for each question are shown in brackets **[ ]**.
- This document has **8** pages.

ADVICE
- Read each question carefully before you start your answer.

Section A (40 marks)

1 (a) The complex number $a + \mathrm{i}b$ is denoted by z.

(i) Write down z^*. **[1]**

(ii) Find $\mathrm{Re}(\mathrm{i}z)$. **[2]**

(b) The complex number w is given by $w = \dfrac{5 + \mathrm{i}\sqrt{3}}{2 - \mathrm{i}\sqrt{3}}$.

(i) In this question you must show detailed reasoning.

Express w in the form $x + \mathrm{i}y$. **[2]**

(ii) Convert w to modulus-argument form. **[2]**

2 In this question you must show detailed reasoning.

Find the angle between the vector $3\mathbf{i} + 2\mathbf{j} + \mathbf{k}$ and the plane $-x + 3y + 2z = 8$. **[5]**

3 (a) Using partial fractions and the method of differences, show that

$$\frac{1}{1 \times 3} + \frac{1}{2 \times 4} + \frac{1}{3 \times 5} + \ldots + \frac{1}{n(n+2)} = \frac{3}{4} - \frac{an + b}{2(n+1)(n+2)},$$

where a and b are integers to be determined. **[5]**

(b) Deduce the sum to infinity of the series.

$$\frac{1}{1 \times 3} + \frac{1}{2 \times 4} + \frac{1}{3 \times 5} + \ldots.$$ **[1]**

4 (a) (i) Given that $\mathrm{f}(x) = \sqrt{1 + 2x}$, find $\mathrm{f}'(x)$ and $\mathrm{f}''(x)$. **[2]**

(ii) Hence, find the first three terms of the Maclaurin series for $\sqrt{1 + 2x}$. **[2]**

(b) Hence, using a suitable value for x, show that $\sqrt{5} \approx \dfrac{143}{64}$. **[2]**

5 **(a) In this question you must show detailed reasoning.**

Determine the sixth roots of -64, expressed in $re^{i\theta}$ form. **[4]**

(b) Represent the roots on an Argand diagram. **[3]**

6 The matrices **M** and **N** are $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$ respectively.

(a) In this question you must show detailed reasoning.

Determine whether **M** and **N** commute under matrix multiplication. **[3]**

(b) Specify the transformation of the plane associated with each of the following matrices.

(i) **M** **[1]**

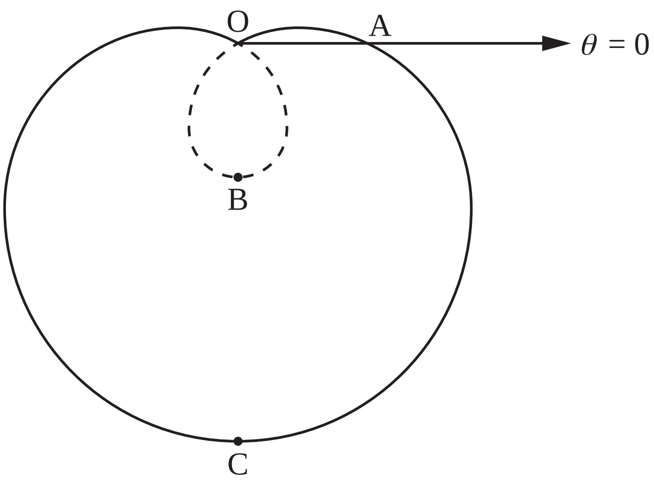
(ii) **N** **[2]**

(c) State the significance of the result in part **(a)** for the transformations associated with **M** and **N**. **[1]**

(d) Use an algebraic method to show that all lines parallel to the x-axis are invariant lines of the transformation associated with **N**. **[2]**

Section B (104 marks)

7 The diagram below shows the curve with polar equation $r = a(1 - 2\sin\theta)$ for $0 \leqslant \theta \leqslant 2\pi$, where a is a positive constant.



The curve crosses the initial line at A, and the points B and C are the lowest points on the two loops.

(a) Find the values of r and θ at the points A, B and C. **[3]**

(b) Find the set of values of θ for the points on the inner loop (shown in the diagram with a broken line). **[3]**

8 Prove by mathematical induction that $8^n - 3^n$ is divisible by 5 for all positive integers n. **[5]**

9 In an electrical circuit, the alternating current I amps is given by $I = a\sin nt$, where t is the time in seconds and a and n are positive constants. The RMS value of the current, in amps, is defined to be the square root of the mean value of I^2 over one complete period of $\frac{2\pi}{n}$ seconds.

Show that the RMS value of the current is $\frac{a}{\sqrt{2}}$ amps. **[6]**

10 The equation $x^3 - 4x^2 + 7x + c = 0$, where c is a constant, has roots α, β and $\alpha + \beta$.

(a) Determine the roots of the equation. **[6]**

(b) Find c. **[1]**

11 Solve the differential equation $\cosh x \frac{\mathrm{d}y}{\mathrm{d}x} - 2y \sinh x = \cosh x$, given that $y = 1$ when $x = 0$. **[7]**

12 Show that $\sin^5\theta = a\sin 5\theta + b\sin 3\theta + c\sin\theta$, where a, b and c are constants to be determined. **[7]**

13 **(a)** On separate Argand diagrams, show the set of points representing each of the following inequalities.

(i) $|z| \leqslant \sqrt{5}$ **[3]**

(ii) $|z + 2 - 4\mathrm{i}| \geqslant |z - 2 - 6\mathrm{i}|$ **[3]**

(b) Show that there is a unique value of z, which should be determined, for which both $|z| \leqslant \sqrt{5}$ and $|z + 2 - 4\mathrm{i}| \geqslant |z - 2 - 6\mathrm{i}|$. **[8]**

14 Three planes have equations

$$\begin{aligned} kx \qquad - \ z &= 2, \\ -x + ky + 2z &= 1, \\ 2kx + 2y + 3z &= 0, \end{aligned}$$

where k is a constant.

(a) By considering a suitable determinant, show that the three planes meet at a point for all values of k. **[5]**

(b) Using a matrix method, find, in terms of k, the coordinates of the point of intersection of the planes. **[8]**

15 **In this question you must show detailed reasoning.**

Evaluate $\displaystyle\int_1^2 \frac{1}{\sqrt{1+2x-x^2}}\,dx$, giving your answer in terms of π. **[5]**

16 The point P (4, 1, 0) is equidistant from the plane $2x+y+2z=0$ and the line $\dfrac{x-3}{2}=\dfrac{y-1}{b}=\dfrac{z+5}{3}$, where $b>0$.

Determine the value of b. **[10]**

17 Two similar species, X and Y, of a small mammal compete for food and habitat. A model of this competition assumes, in a particular area, the following.

- In the absence of the other species, each species would increase at a rate proportional to the number present with the same constant of proportionality in each case.
- The competition reduces the rate of increase of each species by an amount proportional to the number of the other species present.

So if the numbers of species X and Y present at time t years are x and y respectively, the model gives the differential equations

$$\frac{\mathrm{d}x}{\mathrm{d}t} = kx - ay \quad \text{and} \quad \frac{\mathrm{d}y}{\mathrm{d}t} = ky - bx,$$

where k, a and b are positive constants.

(a) **(i)** Show that the general solution for x is $x = A\mathrm{e}^{(k+n)t} + B\mathrm{e}^{(k-n)t}$, where $n = \sqrt{ab}$ and A and B are arbitrary constants. **[6]**

(ii) Hence find the general solution for y in terms of A, B, k, n, a and t. **[2]**

Observations suggest that suitable values for the model are $k = 0.015$, $a = 0.04$ and $b = 0.01$. You should use these values in the rest of this question.

(b) When $t = 0$, the numbers present of species X and Y in this area are x_0 and y_0 respectively.

(i) Show that $x = \frac{1}{2}(x_0 - 2y_0)\mathrm{e}^{0.035t} + \frac{1}{2}(x_0 + 2y_0)\mathrm{e}^{-0.005t}$. **[3]**

(ii) Hence show that $y = \frac{1}{4}(x_0 + 2y_0)\mathrm{e}^{-0.005t} - \frac{1}{4}(x_0 - 2y_0)\mathrm{e}^{0.035t}$. **[1]**

(c) Use initial values $x_0 = 500$ and $y_0 = 300$ with the results in part **(b)** to determine what the model predicts for each of the following questions.

(i) What numbers of each species will be present after 25 years? **[2]**

(ii) **In this question you must show detailed reasoning.**

When will the numbers of the two species be equal? **[4]**

(iii) Does either species ever disappear from the area? Justify your answer. **[3]**

(d) Different initial values will apply in other areas where the two species compete, but previous studies indicate that one species or the other will eventually dominate in any given area.

(i) Identify a relationship between x_0 and y_0 where the model does **not** predict this outcome. **[1]**

(ii) Explain what the model predicts in the long term for this exceptional case. **[2]**

END OF QUESTION PAPER